Vortices, flux tubes and other structures in the Ginzburg-Landau model:

a possible fine structure of the mixed state?

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(February 1, 2008)

Abstract

We present new regular static isolated cylindrically symmetric solutions for the Ginzburg-Landau model which have finite Gibbs free energy. These configurations (which we call the *flux tube* and *type B* solutions) are energetically favorable in the interval of the external magnetic fields between the thermodynamic critical value H_c and the upper critical field H_{c_2} which indicates that they are important new elements of the mixed state describing a transition from vortices to the normal state.

PACS no.: 03.50.-z; 11.15.Kc; 74.20.De; 74.60.-w

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The existence of vortices was predicted by Abrikosov [1] within the framework of the phenomenological Ginzburg-Landau (GL) theory of superconductivity. Nielsen and Olesen [2], noticing the mathematical equivalence of the GL model to the Abelian theory of coupled gauge and Higgs fields, demonstrated the relevance of the vortex (string-like) structures in the high energy particle physics. An isolated vortex is a static regular cylindrically symmetric solution of the classical second-order GL equations with the finite energy (line density), see earlier numerical results in [3]. Our purpose is to present new exact regular solutions with the *finite Gibbs free energy*. We consider isolated structures and thus confine ourselves to the case of cylindrical symmetry.

The Lagrangian of the model [2] is

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|D_\mu\Phi|^2 - \frac{\lambda}{4}\left(|\Phi|^2 - \frac{\mu^2}{\lambda}\right)^2,$$
(1)

where Φ is a complex scalar field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, $D_\mu = \partial_\mu + iA_\mu$, while λ and μ are the coupling constants. Since μ has the dimension of an inverse length, it is convenient to introduce for the cylindrical system (ρ, θ, z) a new (dimensionless) radial coordinate r via $\rho = (\sqrt{\lambda}/\mu)\,r$. Looking for *static* configurations, we use the cylindrically symmetric ansatz $A_2 = f(r)$, $\Phi = (\mu/\sqrt{\lambda})\,\varphi(r)$, where f, φ are two *real* functions. Notice that we do not include the phase factor $e^{in\theta}$ for the scalar field (cf. [2]). The field φ is defined up to a gauge transformation and we find it more convenient to work in such a real gauge. The magnetic field has only one component in z-direction, given by $H = (\mu^2/\lambda)\,h$, where we denote the *dimensionless* magnetic field by $h = f'/r$.

With the above assumptions, the equations of motion reduce to

$$r^2 f'' - rf' = r^2\varphi^2 f,$$
(2)

$$r^2\varphi'' + r\varphi' = \varphi\left(f^2 + \lambda r^2(\varphi^2 - 1)\right).$$
(3)

From (1), the energy per unit length for static cylindrical configurations reads

$$\mathcal{E} = \frac{\pi\mu^2}{\lambda}\int\limits_0^\infty drr\left[h^2 + (\varphi')^2 + \frac{f^2\varphi^2}{r^2} + \frac{\lambda}{2}\left(\varphi^2 - 1\right)^2\right].$$
(4)

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Fields which satisfy (2)-(3) also produce an extremum of $\mathcal{E}$. The latter has an absolute minimum $\mathcal{E} = 0$ for the trivial solution ($\varphi = 1, f = 0$) which describes the Meissner state with superconducting order in all points of a sample and the magnetic field absent inside. The vortex [1–4] presents an example of a nontrivial finite $\mathcal{E}$ solution with the magnetic field penetrating along a thin non-superconducting core around $r = 0$.

Although $\mathcal{E} > 0$ for the vortex, one can prove that it becomes energetically more preferable, above the first critical field H_{c_1}, than the trivial Meissner state by inspecting the Gibbs free energy (per unit length) which is defined by $\mathcal{G} = \mathcal{E} - \int d^2x\,(\boldsymbol{H}\boldsymbol{H}_{\mathrm{ext}})$. We will assume that the external constant magnetic field is also directed along the z-axis, and use the dimensionless variable defined, as above for H, by $H_{\mathrm{ext}} = (\mu^2/\lambda)h_{\mathrm{ext}}$. For the understanding of a transition to the normal state, of particular interest is the functional

$$\Delta\mathcal{G} = \frac{\pi\mu^2}{\lambda} \int\limits_0^\infty dr\, r\left[(h - h_{\mathrm{ext}})^2 \right.$$
$$\left. + (\varphi')^2 + \frac{f^2\varphi^2}{r^2} + \frac{\lambda}{2}\left(\varphi^4 - 2\varphi^2\right) \right], \tag{5}$$

which describes the difference $\mathcal{G} - \mathcal{G}_{nh}$ between the Gibbs free energy of a particular configuration and that of the normal state ($\varphi = 0, h = h_{\mathrm{ext}}$). It is important to notice that (5) has the *same equations of extremals* as the energy functional (4), namely (2)-(3). However, unlike the strictly positive $\mathcal{E}$, the functional (5) can have any sign.

Let us study the regular solutions of the GL equations (2)-(3). At the symmetry axis, we find two types of regularity conditions for f and φ:

(A) Potential f is non-zero while scalar field φ vanishes at the origin,

$$f = N + ar^2 + \frac{1}{4N(N+1)}b^2\,r^{2N+2} + O(r^{2N+4}), \tag{6}$$
$$\varphi = b\,r^N \left(1 + \frac{N}{2(N+1)}\left(a - \frac{\lambda}{2N}\right)r^2 + O(r^4)\right), \tag{7}$$

where $N = \pm 1, \pm 2, \ldots$ and parameters a, b are arbitrary.

(B) Potential f vanishes while scalar field φ is nontrivial at the origin,

$$f = ar^2\left(1 + b^2\,r^2/8 + O(r^4)\right), \tag{8}$$

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$$\varphi = b \left(1 + \lambda(b^2 - 1)r^2/4 + O(r^4)\right), \tag{9}$$

with some parameters a, b.

The vortex solution satisfies (A) and is distinguished by the well known asymptotic conditions at infinity ($\varphi \to 1, f \to 0$). Analysing the (B) case, one can prove that *no finite energy* solutions for type B regularity conditions exist. However, it is possible to find *finite Gibbs free energy* regular type B solutions which satisfy at infinity

$$h(r) = \frac{1}{r} f'(r)\Big|_{r \to \infty} \longrightarrow h_{\text{ext}}, \tag{10}$$

$$\varphi(r)|_{r \to \infty} \longrightarrow 0. \tag{11}$$

The simplest solutions with φ having no nodes are shown in Fig. 1 (in all figures we use $\kappa := \sqrt{\lambda}$). There exist also more nontrivial solutions with nodes. A principal difference of these solutions (which we will call "type B" solutions for brevity) from the vortex solutions with a quantized flux lies in the fact that the magnetic field is now asymptotically constant, and hence the total magnetic flux is infinite. However a reasonable replacement of the flux is provided by

$$M := \int \rho d\rho d\theta (H - H_{\text{ext}}) = 2\pi \int_0^\infty dr r(h - h_{\text{ext}}). \tag{12}$$

Defined formally as a difference of fluxes, this variable is usually interpreted as a *magnetization* per unit volume [4]. Unlike the quantized flux for the vortices, M can have an arbitrary value for the type B solutions.

Numerical integration reveals the following interesting properties of the nodeless type B solutions: For $\lambda > 1/2$ they all have $\Delta\mathcal{G} < 0$, for $\lambda = 1/2$ always $\Delta\mathcal{G} = 0$, and $\Delta\mathcal{G} > 0$ for $\lambda < 1/2$. Hence, for an ideal type II superconductor the type B configuration is energetically more preferable than the normal state. However, the type B solutions do not exist for every value of the external field h_{ext}. For a given $\lambda > 1/2$, one finds $h_{\text{ext}} \leq \lambda$ for all solutions, and thus the upper limit is exactly equal to the *second critical field* $h_{c_2} = \lambda$.

However, this is not the end of the story. Besides the type B configurations, there are other nontrivial solutions with finite values of the Gibbs functional (5). Namely, let us

take the type A regularity conditions (6)-(7) at the origin, while at infinity we consider the asymptotics (10)-(11), which suggests a possible physical interpretation of such solutions as emerging from a "gluing" of a vortex with a type B configuration. These solutions are shown in Fig. 2. Their form explains why we call them *flux tubes* [5]: There is a non-superconducting core filled by the magnetic field (looking like a vortex), surrounded by a superconducting tube (almost completely free of the magnetic field). Outside such a tube the sample quickly reduces to a normal state ($\varphi = 0$) with a constant external field in it.

Each family of the flux tube (FT) solutions has two branches. For the nodeless $\lambda = 1$ case, we find that one of these branches is characterized by the positive Gibbs free energy, and another has negative Gibbs free energy. However, for $\lambda > 9/2$ both branches describe configurations with $\Delta\mathcal{G} < 0$. We find it convenient to depict these branches in the form of magnetization curves, see Fig. 3. For $\lambda \leq 1/2$, both branches have $\Delta\mathcal{G} > 0$.

Similarly to the type B configurations, we find the flux tube solutions with nodes. For $\lambda = 1$, all the FT solutions with one node have $\Delta\mathcal{G} > 0$. However, with increasing λ this changes. The crucial point is the position of the thermodynamic critical value $h_c = \sqrt{\lambda/2}$ relative to the "limiting values" of h_{ext} at which a FT family has vanishing magnetization and Gibbs free energy. Numerically, we discovered that these limiting points are located, on the h_{ext} axis, at $\lambda, \frac{1}{3}\lambda, \frac{1}{5}\lambda, \frac{1}{7}\lambda, \ldots$. Below we explain these values with the help of the linearization analysis.

Since both, the type B and the FT solutions, have a well defined finite Gibbs free energy, it is natural to compare them. We find that always $\Delta\mathcal{G}_{\text{FT}} < \Delta\mathcal{G}_{\text{B}}$. For illustration, we display $\Delta\mathcal{G}$ as a function of h_{ext} for $\lambda = (2.25)^2$ in Fig. 4 which shows that *the flux tube configurations are energetically more preferable*. This holds true for any λ.

The best way to understand the structure of the type B and FT solutions in the limit of vanishing magnetization ($M \to 0$) and Gibbs free energy ($\Delta\mathcal{G} \to 0$) is to study the linearized GL equations. The linearization of the system (2)-(3) leads to the Schrödinger-type equation for φ with the potential of a circular oscillator. Regular solutions exist only when

$$h = \frac{s\lambda}{1 + 2n + sN + |N|}, \tag{13}$$

where $n = 0, 1, 2, ...$ and $s = \pm 1$. Corresponding eigenfunctions $\varphi_{n,N}$ are given in terms of the Laguerre polynomials, with n equal to the number of their zeros (nodes). Let us denote $h_k := \lambda/(2k + 1), k = 0, 1, 2, ...$ For $n = 0$ and $sN = -|N|$, one finds that the maximal eigenvalue of (13) is $h = s\lambda = sh_0$, which is precisely the second critical field $h_{c_2} = \lambda$. At the first sight, this fact may seem to be well known in the literature on the type II superconductors, cf. [1,4]. However, the peculiar point is the conclusion that the vortices apparently have nothing to do with the determination of h_{c_2}. As we see, the latter is determined by the existence condition for the nodeless type B and FT solutions which are energetically more preferable than the normal state. It is clear that the eigenfunctions $\varphi_{0,N} = r^{|N|} \exp(-\frac{\lambda}{4} r^2)$ describe the linearized regular type B (for $N = 0$) and FT (for $N \neq 0$) solutions. It seems worthwhile to notice also that the starting point for the well known (see, e.g., [1,4]) construction of a vortex lattice near h_{c_2} is not the vortex, as one might think, but a linearized type B solution.

The physical meaning of the lower eigenvalues (13) is now also clear: these define the limiting values for the external magnetic field at which the exact type B and FT solutions become "linearizable" and thus disappear with vanishing magnetization and Gibbs free energy. This confirms our numerical observations: For instance, both the nodeless FT and the type B configurations have the limiting magnetic field values $h_1 = \lambda/3$ and $h_0 = h_{c_2}$, while the FT and the type B with one node "live" between $h_2 = \lambda/5$ and $h_1 = \lambda/3$. In general, the limiting points for the solutions with k nodes are h_{k+1} and h_k. As we already mentioned, for a given λ the position of the thermodynamic critical value $h_c = \sqrt{\lambda/2}$ relative to the interval $[h_{k+1}, h_k]$ is important. If h_c belongs to this interval, then $h_{k+1} < h_{ext} < h_k$ for all solutions in this family. However when h_c does not belong to this interval, then $[h_{k+1}, h_k]$ is extended up to h_c. In particular, we find that for every FT family, the magnetization diverges when $h_{ext} \to h_c$. Moreover, one can show that the Gibbs free energy for a vortex and for a flux tube solution are equal for $h_{ext} = h_c$, [7].

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One may further weaken the conditions at infinity, requiring $f \to 0$ for the potential, but for the scalar field simply demanding regularity and finiteness. This gives another class of regular configurations which we call the oscillating solutions (OS). For large r we can neglect the nonlinear terms in (3), and get for the scalar field the linearized equation $\varphi'' + \varphi'/r + \lambda\varphi = 0$. Its solution is the Bessel function $\varphi = J_0(\sqrt{\lambda}r)$ which is confirmed by the direct numerical integration. Such OS seem to be relevant to observations made in [6,7]. In our opinion these new solutions are unstable configurations preceding to the completely formed Abrikosov vortices. Notice, that an oscillating character of such solutions may resemble an "intermediate" superconductor state with coexisting normal and superconducting regions. But a big difference is that the magnetic field penetrates only at the center, and the magnetic flux is quantized precisely as for the vortices. Oscillating solutions exist both for the type A and B conditions, in the latter case the magnetic field is completely zero. A curious feature is that for a finite sample in absence of an external magnetic field an oscillating state is energetically more preferable than a purely normal state.

In each class of solutions the decisive role is played by the values of the parameters (a, b) appearing in the regularity conditions (6)-(7) and (8)-(9). Numerically, all the reported new solutions were obtained after a "fine tuning" of these parameters. Let us draw a kind of a "phase diagram" on the (a, b) plane showing explicitly the domains of existence for different solutions. Since the vortices, flux tubes, and oscillating solutions all satisfy the type A conditions (6)-(7), we can display them on the same (a, b)-plane, see Fig. 5 for the case of $\lambda = 1$. The big dot shows the "position" of a vortex, while each curve represents a complete family of solutions (a point on a curve gives (a, b) for a particular solution). The curves hit the a-axis ($b = 0$) at one of the possible limiting values: $a = \pm\frac{1}{2}h_k, k = 0, 1, \ldots$. Curves for FT with increasing number of nodes are "concentrating" around the oscillating curve which seems to indicate that the oscillating solutions are unstable and a small perturbation may cause their decay into a nearby flux tube with a finite number of nodes. Moving along any FT curve away from the a-axis, one hits the vortex dot, where magnetization diverges.

In this paper we have presented new numerical solutions of the cylindrically symmetric

GL equations. Noticing that the latter are common extremals for the energy (4) and the Gibbs free energy (5) functionals, we describe the regular solutions with *finite Gibbs free energy*. We show that h_{c_2} is not the magnetic field below which the vortex becomes energetically more preferable than the normal state, but it is the field at which the formation of the type B and flux tube structures starts, since $\Delta\mathcal{G}_{\text{FT,B}} < 0$ for $h_{\text{ext}} < h_{c_2}$. Our results show that the flux tube solutions without node remain energetically most preferable from h_{c_2} down to h_c, after which the vortices become energetically dominant. It is worthwhile to stress, that our results do not contradict the previous knowledge about the mixed state in the type II superconductors. Notice that, after all, one can interpret a flux tube solution as a vortex "surrounded" by a type B configuration. In a certain sense, one can speak of a "fine structure" for the mixed state: Different configurations (vortices, type B and FT solutions with k nodes, $k = 0, 1, \ldots$) can exist for any external field h_{ext} between h_{c_1} and h_{c_2} (cf. Table I).

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FIGURES

FIG. 1. Type B solutions without node for $\kappa = 1.0$: The scalar field φ and the magnetic field h with initial values $\varphi(0) = 0.9, 0.6, 0.3$ and $f(0) = 0.0$.

FIG. 2. Flux tube solutions without node for $\kappa = 0.5, 1.0, 1.5$ with the fixed value of magnetization.

FIG. 3. Magnetization curves for nodeless FT with $\kappa = 0.5, 1.0, 1.5$. For each κ, the corresponding limiting values h_1, h_2, h_c, and h_{c_2} are drawn. The magnetization diverges at h_c.

FIG. 4. $\Delta\mathcal{G}$ against the external magnetic field for the flux tubes and the type B solutions with $\kappa = 2.25$. The flux tubes have lower $\Delta\mathcal{G}$ and thus are energetically more preferable.

FIG. 5. (a, b)-diagram for $\kappa = 1.0$. The big dot describes the corresponding vortex solution. The drawn, broken and dotted lines represent the nodeless flux tubes, flux tubes with one node, and oscillating solutions, respectively.
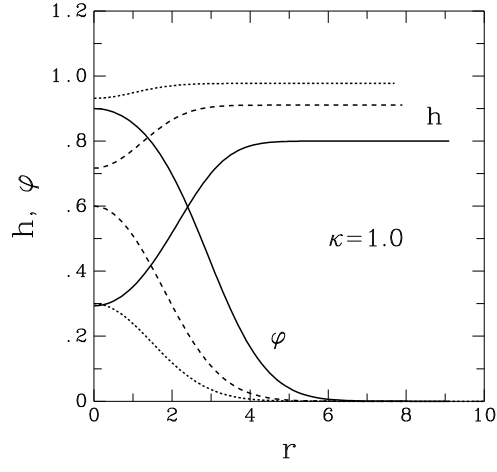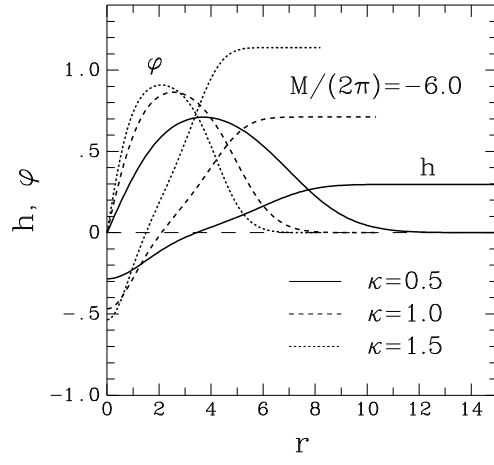
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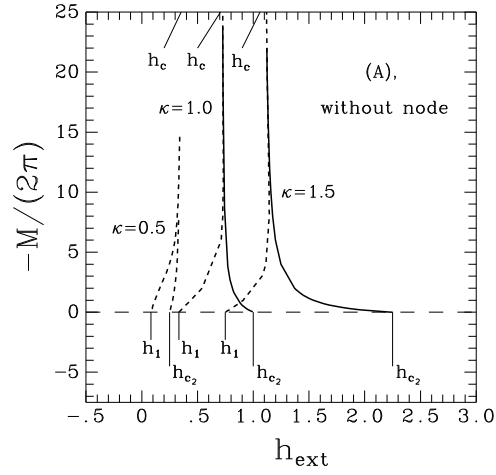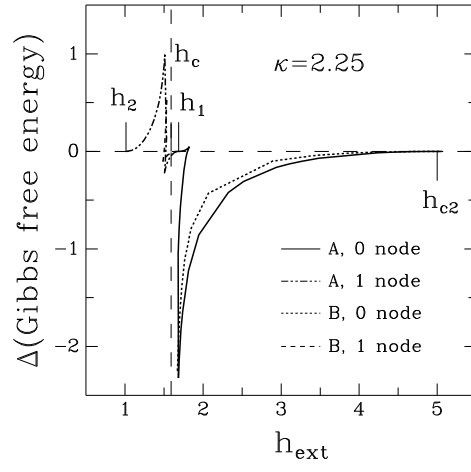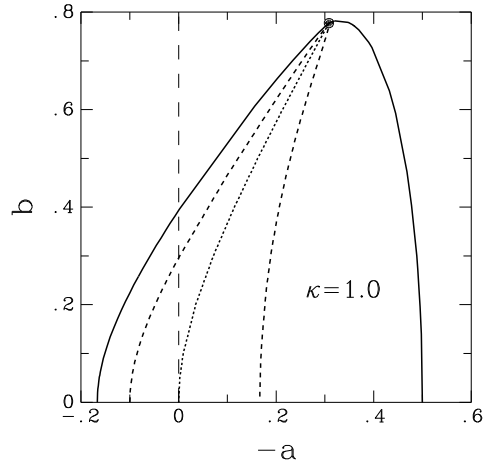

FIGURE 2:

FIGURE 3:

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FIGURE 4:

FIGURE 5:

TABLES

TABLE I. Existence domains of different GL solutions: fine structure of the mixed state?

region	energetically preferable
$h_{c_3} < h_{\text{ext}}$	normal
$h_{c_2} < h_{\text{ext}} < h_{c_3}$	surface superconductivity
$h_1 < h_{\text{ext}} < h_{c_2}$	0-node FT
$h_2 < h_{\text{ext}} < h_1$	0-node FT, 1-node FT
$h_3 < h_{\text{ext}} < h_2$	0-node FT, 1-node FT, 2-node FT
$h_{c_1} < h_{\text{ext}} < h_c$	pure vortices
$0 < h_{\text{ext}} < h_{c_1}$	Meissner